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On Idempotent and Hyperassociative Structures

Yu. Movsisyan[1,2*] and M. Yolchyan[1]**

(Submitted by S. A. Grigoryan)

[1] *Yerevan State University, Yerevan, 0025 Armenia*

[2] *University of Bergen, P.O. Box 7800 5020, Bergen, Norway*

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Abstract—The paper is devoted to the study of the structures of idempotent and hyperassociative algebras. The goal is to explain new methodological developments in algebras, which will be of growing importance in the second order logic. Our results extend the corresponding results on semigroups too.

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1. INTRODUCTION

The algebra $(Q;\Sigma)$ is called idempotent, if each its operation is idempotent. If the groupoid $Q(\cdot)$ is idempotent, then the operation $\cdot$ is also called idempotent. The semigroup $Q(\cdot)$ is called semilattice, if it is idempotent and commutative. If $Q(\cdot)$ is a semilattice, then the operation $\cdot$ is called semilattice operation too. An algebra with binary operations is called a binary algebra, [1].

A binary algebra $(Q;\Sigma)$ is called:

1) anticommutative if it satisfies the following condition: $X(x,y) = X(y,x) \Rightarrow x = y$, where $X \in \Sigma$ and $x, y \in Q$;

2) with the transitive commutativity property, if it satisfies the following condition:

$$X(x,y) = X(y,x) \& X(y,z) = X(z,y) \Rightarrow X(x,z) = X(z,x),$$

where $X \in \Sigma$ and $x, y, z \in Q$.

For the second order formulae (and the second order languages) see [2–4]. Let us recall, that a hyperidentity [5–10] (or $\forall(\forall)$-identity) is a second-order formula of the following form:

$$\forall X_1, \ldots, X_m \quad \forall x_1, \ldots, x_n (\omega_1 = \omega_2),$$

where ω_1, ω_2 are words (terms) in the alphabet of functional variables $X_1, \ldots, X_m$ and objective variables $x_1, \ldots, x_n$. However hyperidentities are usually presented without universal quantifiers: $\omega_1 = \omega_2$. The hyperidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra $(Q;\Sigma)$ if this equality holds whenever every object variable x_j is replaced by an arbitrary element from Q and every functional variable X_i is replaced by an arbitrary operation of the corresponding arity from Σ. A possibility of such replacement is supposed, that is

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q;\Sigma)} = T_{(\Sigma)},$$

where $|S|$ is the arity of S, and $T_{(Q;\Sigma)}$ is called the arithmetic type of $(Q;\Sigma)$. A T-algebra is an algebra with arithmetic type $T \subseteq N$. A class of algebras is called a class of T-algebras if every algebra in it is a T-algebra.

*E-mail: yurimovsisyan@yahoo.com, movsisyan@ysu.am

**E-mail: marlen.yolchyan94@gmail.com

A coidentity [5–10] (or $(\exists)\forall$-identity) is a second-order formula of the following form:

$$\exists x_1, \dots, x_n \quad \forall X_1, \dots, X_m (w_1 = w_2).$$

A coidentity is usually presented without quantifiers too: $\omega_1 = \omega_2$. The coidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra $(Q; \Sigma)$, if there exist values for object variables $x_1, \dots, x_n$ from Q, such that the equality $\omega_1 = \omega_2$ holds when every functional variable in it is replaced by any operation of the corresponding arity from Σ (a possibility of such replacement is supposed). In addition the object variables in the notation of the coidentity $\omega_1 = \omega_2$ are replaced by corresponding fixed values from Q.

Examples. 1) In any multioperator Ω-group the following coidentity is valid:

$$X(\underbrace{0, \dots, 0}_{n}) = 0,$$

for all $n \in T_{(\Omega)}$, where all object variables are replaced by the zero element of the Ω-group, [11].

2) (J. von Neumann) Let $L(+, \cdot)$ be a modular lattice and $a, b, c \in L$. The sublattice of L, generated by the elements a, b, c, will be distributive if the following coidentity of left distributivity

$$X\left(a, Y(b, c)\right) = Y\left(X(a, b), X(a, c)\right)$$

holds in $L(+, \cdot)$.

The algebra $(Q; \Sigma)$ is idempotent, if the following hyperidentity of idempotency

$$X(\underbrace{x, \dots, x}_{n}) = x \qquad \text{(id)}$$

is valid for all $n \in T_{(Q;\Sigma)}$.

The binary algebra $(Q; \Sigma)$ is called hyperassociative, if it satisfies the following hyperidentity of associativity:

$$X(x, Y(y, z)) = Y(X(x, y), z). \qquad (\text{as}_1)$$

The binary algebra $(Q; \Sigma)$ is called rectangular, if it satisfies the following hyperidentity of rectangularity:

$$X(x, X(y, x)) = x. \qquad \text{(rect)}$$

We say, that

a) the algebra $(Q; \Sigma)$ is a rectangular structure of semilattices, if there exist a congruence q of $(Q; \Sigma)$, such that the corresponding quotient algebra is rectangular and any operation of Σ is semilattice operation on any equivalence class of q;

b) the algebra $(Q; \Sigma)$ is a semilattice structure of rectangular semigroups, if there exist a congruence q of $(Q; \Sigma)$, such that any operation in the corresponding quotient algebra is a semilattice operation and any operation of Σ is rectangular operation on any equivalence class of q (cf. [10, 12]).

The algebra $(Q; \Sigma)$ is called functionally trivial, if the set of its n-ary operations is singleton for any $n \in T_{(Q;\Sigma)}$.

If $(Q; \Sigma)$ is a functionally non-trivial idempotent algebra with the hyperidentity of associativity (as_1), then the cardinality $|Q| \geq 4$. An example of an idempotent functionally non-trivial algebra $Q(+, \cdot)$ with hyperidentity of associativity (as_1) is given by the Cayley tables of their operations $+$ and $\cdot$ below:

$+$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	2	1	4
3	3	3	3	3
4	4	4	4	4

$\cdot$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	2	4	4
3	3	3	3	3
4	4	4	4	4

Moreover, there exist 24 functionally non-trivial idempotent algebras $Q(+, \cdot)$, $|Q| = 4$, with the hyperidentity of associativity (as_1), [8]. Further, the superproduct of two such algebras is an idempotent algebra with four binary operations, satisfying the hyperidentity of associativity (as_1), [9].

The present paper is devoted to the study of structure of idempotent and hyperassociative algebras.

2. THE CONGRUENCE θ^* AND θ

Consider the idempotent and hyperassociative algebra $(Q;\Sigma)$. On the algebra $(Q;\Sigma)$ define the following four relations:

$$\theta = \{(x,y) \in Q \times Q | X(x,y) = X(y,x),\ \forall X \in \Sigma\},$$

$$\theta^* = \{(x,y) \in Q \times Q | X(x, X(y,x)) = x, X(y, X(x,y)) = y,\ \forall X \in \Sigma\},$$

$$\theta_1 = \{(x,y) \in Q \times Q | X(x,y) = y, X(y,x) = x,\ \forall X \in \Sigma\},$$

$$\theta_2 = \{(x,y) \in Q \times Q | X(x,y) = x, X(y,x) = y,\ \forall X \in \Sigma\}.$$

Lemma 1. *The relations θ_1, θ_2 are equivalence relations.*

Proof. According to the definition of the equivalence relation, in order to prove the lemma it is necessary to verify the following conditions:

1. $(x,x) \in \theta_i, i = 1,2,$
2. $(x,y) \in \theta_i \Rightarrow (y,x) \in \theta_i, i = 1,2,$
3. $(x,y) \in \theta_i \& (y,z) \in \theta_i \Rightarrow (x,z) \in \theta_i, i = 1,2.$

The first and second conditions of equivalence relation follow from the definitions of the relations θ_1, θ_2. It remains to check the third condition.

For $i = 1$, we have, $(x,y) \in \theta_1 \Rightarrow X(x,y) = y \& X(y,x) = x$, and $(y,z) \in \theta_1 \Rightarrow X(y,z) = z \& X(z,y) = y$.

$$X(x,z) \overset{X(y,z)=z}{=} X(x, X(y,z)) \overset{(as_1)}{=} X(X(x,y),z) \overset{X(x,y)=y}{=} X(y,z)$$
$$= z \& X(z,x) \overset{X(y,x)=x}{=} X(z, X(y,x)) \overset{(as_1)}{=} X(X(z,y),x) \overset{X(z,y)=y}{=} X(y,x) = x \Rightarrow (x,z) \in \theta_1.$$

For $i = 2$, we can check the third condition similarly. □

Lemma 2. *The following property is valid:*

$$(x,y) \in \theta^* \Rightarrow (x, X(x,y)) \in \theta_1 \& (x, X(y,x)) \in \theta_2, \quad \forall X \in \Sigma.$$

Proof. We have

$$(x,y) \in \theta^* \Rightarrow X(x, X(y,x)) = x \& X(y, X(x,y)) = y, \quad \forall X \in \Sigma.$$

$$Y(x, X(x,y)) \overset{(as_1)}{=} X(Y(x,x),y) \overset{(id)}{=} X(x,y), \tag{1}$$

$$Y(X(x,y),x) \overset{(as_1)}{=} X(x, Y(y,x)) \overset{Y(x,Y(y,x))=x}{=} X(Y(x,Y(y,x)), Y(y,x))$$
$$\overset{(as_1)}{=} Y(x, X(Y(y,x), Y(y,x))) \overset{(id)}{=} Y(x, Y(y,x)) = x. \tag{2}$$

From (1) and (2) it follows, that $(x, X(x,y)) \in \theta_1$.

The second part of the assertion we can check similarly. □

Lemma 3. *The following properties are valid:*

$$(x,y) \in \theta_1 \Rightarrow (X(z,x), X(z,y)) \in \theta_1, \quad \forall z \in Q, \quad \forall X \in \Sigma, \tag{3}$$

$$(x,y) \in \theta_2 \Rightarrow (X(x,z), X(y,z)) \in \theta_2, \quad \forall z \in Q, \quad \forall X \in \Sigma. \tag{4}$$

Proof. Check the property (3).

We have $(x,y) \in \theta_1 \Rightarrow X(x,y) = y \& X(y,x) = x$, for all $X \in \Sigma$. Take $z \in Q$, $X, Y \in \Sigma$ and show, that $Y(X(z,x), X(z,y)) = X(z,y)$ and $Y(X(z,y), X(z,x)) = X(z,x)$, which, according to the definition, means, that $(X(z,x), X(z,y)) \in \theta_1$;

$$Y(X(z,x), X(z,y)) \overset{X(x,y)=y}{=} Y(X(z,x), X(z, X(x,y)))$$

$$\overset{(as_1)}{=} Y(X(z,x), X(X(z,x),y)) \overset{(as_1)}{=} X(Y(X(z,x), X(z,x)), y)$$
$$\overset{(id)}{=} X(X(z,x),y) \overset{(as_1)}{=} X(z, X(x,y)) \overset{X(x,y)=y}{=} X(z,y) \Rightarrow Y(X(z,x), X(z,y)) = X(z,y).$$

In the same way, we can check, that $Y(X(z,y), X(z,x)) = X(z,x)$. That is $(X(z,x), X(z,y)) \in \theta_1$.

The property (4) can be proved similarly. □

Lemma 4. *The following property is valid:*

$$(x,y) \in \theta_i \Rightarrow \begin{cases} (x,y) \in \theta^*, & (5) \\ (X(z,x), X(z,y)) \in \theta^*, \quad \forall z \in Q, \quad \forall X \in \Sigma, & (6) \\ (X(x,z), X(y,z)) \in \theta^*, \quad \forall z \in Q, \quad \forall X \in \Sigma, & (7) \end{cases}$$

where $i = 1, 2$.

Proof. For $i = 1$, we have: $(x,y) \in \theta_1 \Rightarrow X(x,y) = y \& X(y,x) = x$, for all $X \in \Sigma$.

$$X(x, X(y,x)) \overset{X(y,x)=x}{=} X(x,x) = x, \tag{8}$$

$$X(y, X(x,y)) \overset{X(x,y)=y}{=} X(y,y) = y. \tag{9}$$

From (8) and (9) it follows, that $(x,y) \in \theta^*$.

By Lemma 3, we have $(X(z,x), X(z,y)) \in \theta_1$, $\forall z \in Q$, $\forall X \in \Sigma$. Therefore, according to (5), $(X(z,x), X(z,y)) \in \theta^*$, $\forall z \in Q, \forall X \in \Sigma$.

We can check (7), using (6) and the hyperidentites (id), (as_1). For $i = 2$ the actuality of property we can check using Lemma 3. □

Lemma 5. *The following property is valid:* $(x,y) \in \theta^* \cap \theta \Rightarrow x = y$.

Proof. Let $(x,y) \in \theta^* \cap \theta$. That means, the following coidentities hold:

1. $X(x,y) = X(y,x)$,
2. $X(x, X(y,x)) = x$,
3. $X(y, X(x,y)) = y$.

$$x \overset{(2)}{=} X(x, X(y,x)) \overset{(1)}{=} X(x, X(x,y)) \overset{(as_1)}{=} X(X(x,x),y) \overset{(id)}{=} X(x,y)$$
$$\overset{(1)}{=} X(y,x) \overset{(id)}{=} X(X(y,y),x) \overset{(as_1)}{=} X(y, X(x,y)) \overset{(3)}{=} y \Rightarrow x = y.$$

□

Lemma 6. *The following property is valid:* $(x,y) \in \theta_1 \& (y,z) \in \theta_2 \Rightarrow (x,z) \in \theta^*$.

Proof. Let (x,y) $\in \theta_1$& (y,z) $\in \theta_2$. That means, the following coidentities hold:

1. $X(x,y) = y$,
2. $X(y,x) = x$,
3. $X(y,z) = y$,
4. $X(z,y) = z$.

According to the definition, to make sure $(x,z)\in\theta^*$, it is necessary to prove that the coidentities $X(x,X(z,x))=x$, $X(z,X(x,z))=z$ hold;

$$X(x,X(z,x))\overset{(2)}{=}X(X(y,x),X(y,z))\overset{\text{Lemma 3}}{=}X(y,x)\overset{(2)}{=}x\Rightarrow X(x,X(z,x))=x,$$

$$X(z,X(x,z))\overset{(as_1)}{=}X(X(z,x),z)\overset{(4)}{=}X(X(z,x),X(z,y))\overset{\text{Lemma 3}}{=}X(z,y)\overset{(4)}{=}z\Rightarrow X(z,X(x,z)).$$

□

Lemma 7. *If the relation θ^* is transitive, then*

$$(x,y)\in\theta^*\Rightarrow\begin{cases}(X(z,x),X(z,y))\in\theta^* & \forall z\in Q,\quad \forall X\in\Sigma,\\ (X(x,z),X(y,z))\in\theta^* & \forall z\in Q,\quad \forall X\in\Sigma.\end{cases}$$

Proof. We have

$$\begin{aligned}(x,y)\in\theta^*\Rightarrow(x,X(x,y))\in\theta_1,\quad \forall X\in\Sigma \overset{\text{Lemma 4}}{\Rightarrow}(Y(z,x),Y(z,X(x,y)))\in\theta^*\\ \&(Y(x,z),Y(X(x,y),z))\in\theta^*,\quad \forall z\in Q,\quad \forall X,Y\in\Sigma,\end{aligned}\tag{10}$$

$$(x,y)\in\theta^*\overset{def}{\Rightarrow}(y,x)\in\theta^*\overset{\text{Lemma 2}}{\Rightarrow}(y,X(x,y))\in\theta_2,\quad \forall X\in\Sigma$$

$$\overset{\text{Lemma 4}}{\Rightarrow}(Y(z,y),Y(z,X(x,y)))\in\theta^*\&(Y(y,z),Y(X(x,y),z))\in\theta^*,\quad \forall z\in Q,\quad \forall X,Y\in\Sigma.\tag{11}$$

By (10), (11) and the transitivity of θ^* we will get,

$$(X(z,x),X(z,y))\in\theta^*\&(X(x,z),X(y,z))\in\theta^*,\quad \forall z\in Q,\quad \forall X\in\Sigma.$$

□

Lemma 8. *Consider the following hyperidentities:*

$$X(x,x)=x,\tag{12}$$
$$X(x,Y(y,z))=Y(X(x,y),z),\tag{13}$$
$$X(Y(x,y),Y(z,x))=X(Y(x,z),Y(y,x)),\tag{14}$$
$$X(Y(x,y),X(z,x))=X(Y(x,z),X(y,x)),\tag{15}$$
$$X(X(x,y),Y(z,x))=X(X(x,z),Y(y,x)).\tag{16}$$

If the algebra $(Q;\Sigma)$ satisfies the hyperidentities (12)−(15), then it satisfies also the hyperidentity (16).

Proof. We have

$$\begin{aligned}X(X(x,y),Y(z,x))&\overset{(13)}{=}X(x,X(y,Y(z,x)))\overset{(13)}{=}X(x,Y(X(y,z),x))\\ &\overset{(12)}{=}X(Y(x,x),Y(X(y,z),x))\overset{(14)}{=}X(Y(x,X(y,z)),Y(x,x))\\ &\overset{(15)}{=}X(Y(x,X(y,z)),x)\overset{(13)}{=}X(X(Y(x,y),z),x)\overset{(13)}{=}X(Y(x,y),X(z,x))\\ &\overset{(15)}{=}X(Y(x,z),X(y,x))\overset{(13)}{=}X(X(Y(x,z),y),x)\overset{(12)}{=}X(X(Y(x,z),y),Y(x,x))\\ &\overset{(13)}{=}X(Y(x,X(z,y)),Y(x,x))\overset{(14)}{=}X(Y(x,x),Y(X(z,y),x))\\ &\overset{(12)}{=}X(x,Y(X(z,y),x))\overset{(13)}{=}X(x,X(z,Y(y,x)))\overset{(13)}{=}X(X(x,z),Y(y,x)).\end{aligned}$$

□

Lemma 9. *Let $(Q;\Sigma)$ is an idempotent, hyperassociative and rectangular algebra. Then the algebra $(Q;\Sigma)$ is functionally trivial. Therefore, $(Q;\Sigma)$ will be anticommutative semigroup.*

Proof. Take $X,Y\in\Sigma$ and $x,y\in Q$. For prove the lemma, enough to show, that $X(x,y)=Y(x,y)$.

$$X(x,Y(x,y))\overset{(as_1)}{=}Y(X(x,x),y)$$

$$\stackrel{(rect)}{=} Y(X(x,x),X(y,X(x,y))) \stackrel{(as_1)}{=} Y(X(x,x),X(X(y,x),y))$$
$$\stackrel{(as_1)}{=} X(Y(X(x,x),X(y,x)),y) \stackrel{(as_1)}{=} X(X(x,Y(x,X(y,x))),y)$$
$$\stackrel{(as_1)}{=} X(X(x,X(Y(x,y),x)),y) \stackrel{(rect)}{=} X(x,y).$$

On the other hand $X(x,Y(x,y)) \stackrel{(as_1)}{=} Y(X(x,x),y) \stackrel{(id)}{=} Y(x,y)$.

We get $X(x,Y(x,y)) = X(x,y)$, and $X(x,Y(x,y)) = Y(x,y)$, therefore $X(x,y) = Y(x,y)$. □

3. THE STRUCTURE OF IDEMPOTENT AND HYPERASSOCIATIVE ALGEBRAS

Theorem 1. *Let $(Q;\Sigma)$ is an idempotent and hyperassociative algebra. Then the relation*

$$\theta^* = \{(x,y) \in Q \times Q | X(x,X(y,x)) = x, X(y,X(x,y)) = y, \quad \forall X \in \Sigma\}$$

is a congruence defined on the algebra $(Q;\Sigma)$, furthermore, each operation of the corresponding quotient algebra is semilattice operation and the equivalence classes are rectangular (and idempotent) semigroups.

Proof. Show, that the relation θ^* is congruence, each operation of the quotient algebra of that congruence is semilattice operation and the equivalence classes are rectangular semigroups.

By the definition of θ^* and (id), we get the conditions of reflexivity and symmetry of θ^*. Let us check the condition of transitivity:

$$(x,y) \in \theta^* \& (y,z) \in \theta^* \stackrel{\text{Lemma 2}}{\Rightarrow} (y,X(y,x)) \in \theta_1, \quad \forall X \in \Sigma \&$$
$$(y,X(x,y)) \in \theta_2, \quad \forall X \in \Sigma \& (y,X(y,z)) \in \theta_1, \quad \forall X \in \Sigma \&$$
$$(y,X(z,y)) \in \theta_2, \quad \forall X \in \Sigma.$$

By this result and the transitivity of equivalence relations θ_1, θ_2, we get:

$$(X(y,x),X(y,z)) \in \theta_1, \quad \forall X \in \Sigma \& (X(x,y),X(z,y)) \in \theta_2, \quad \forall X \in \Sigma$$
$$\stackrel{\text{Lemma 3}}{\Rightarrow} (Y(z,X(y,x)),Y(z,X(y,z))) \in \theta_1, \quad \forall X,Y \in \Sigma \&$$
$$(X(Y(x,y),x),X(Y(z,y),x)) \in \theta_2, \quad \forall X,Y \in \Sigma$$
$$\stackrel{(as_1)}{\Rightarrow} (X(Y(z,y),x),Y(z,X(y,z))) \in \theta_1, \quad \forall X,Y \in \Sigma \&$$
$$(Y(x,X(y,x)),X(Y(z,y),x)) \in \theta_2, \quad \forall X,Y \in \Sigma$$
$$\stackrel{\text{Lemma 6}}{\Rightarrow} (Y(z,X(x,z)),Y(x,X(y,x))) \in \theta^*.$$

Now, show, that from the coidentity $X(x,X(y,x)) = x$ implies the coidentity $X(x,Y(y,x)) = x$. Indeed,

$$X(x,Y(y,x)) \stackrel{Y(x,Y(y,x))=x}{=} X(Y(x,Y(y,x)),Y(y,x))$$
$$\stackrel{(as_1)}{=} Y(x,X(Y(y,x),Y(y,x))) \stackrel{(id)}{=} Y(x,Y(y,x)) = x.$$

Thus, we have, that it follows from the coidentity $X(z,X(x,z)) = z$ to the coidentity $X(z,Y(x,z)) = z$, i.e., we get

$$(Y(z,X(x,z)),Y(x,X(y,x))) = (x,z) \in \theta^*,$$

that is the transitivity of θ^* is proved.

Now, check the fourth condition of the congruence.

Let $(x,y) \in \theta^* \& (z,t) \in \theta^*$. Show $(X(x,z),X(y,t)) \in \theta^*, \forall X \in \Sigma$.

$$(x,y) \in \theta^* \& (z,t) \in \theta^* \stackrel{\text{Lemma 7}}{\Rightarrow} (X(x,t),X(y,t)) \in \theta^* \& (X(x,z),X(x,t)) \in \theta^*, \quad \forall X \in \Sigma.$$

From here, according to the transitivity of θ^*, we have

$$(X(x,z), X(y,t)) \in \theta^*, \quad \forall X \in \Sigma,$$

i.e. θ^* is a congruence.

Consider the quotient algebra $(D/\theta^*; \Sigma^*)$. Denote the equivalence class by θ^*_x, and the operations of the quotient algebra define like $X^*(\theta^*_x, \theta^*_y) = \theta^*_{X(x,y)}$. The correctness of this definition follows from the congruence condition.

Now, show, that each operation of the quotient algebra $(Q/\theta^*; \Sigma^*)$ is a semilattice operation, i.e. check the following hyperidentities:

1. $X^*(\theta^*_x, \theta^*_x) = \theta^*_x$,
2. $X^*(\theta^*_x, \theta^*_y) = X^*(\theta^*_y, \theta^*_x)$.

Indeed,

$$X^*(\theta^*_x, \theta^*_x) \overset{def}{=} \theta^*_{X(x,x)} \overset{(id)}{=} \theta^*_x \Rightarrow X^*(\theta^*_x, \theta^*_x) = \theta^*_x, \quad X^*(\theta^*_x, \theta^*_y) \overset{def}{=} \theta^*_{X(x,y)}, \quad X^*(\theta^*_y, \theta^*_x) \overset{def}{=} \theta^*_{X(y,x)}.$$

We can show, that $\theta^*_{X(x,y)} = \theta^*_{X(y,x)}$ using the hyperidentities (id) and (as$_1$).

To complete the proof, it remains to show, that for any $x \in Q$, the class θ^*_x is an idempotent and rectangular semigroup.

Take $z, y \in \theta^*_x$, $X \in \Sigma$. According to the definition, θ^*_x, $(x,y) \in \theta^*$, $(x,z) \in \theta^*$. By the transitivity of θ^* we get

$$(y,z) \in \theta^* \Rightarrow X(z, X(y,z)) = z.$$

Now, show, that, for any $x \in$ Q, θ^*_x is closed for all $X \in \Sigma$.

Take $z, y \in \theta^*_x$, $X \in \Sigma$. We have $(y,x) \in \theta$ and $(z,x) \in \theta$, therefore, according to the fourth condition of the congruence, $(X(z,y), X(x,x)) \in \theta$, and since $X(x,x) = x$, then $(X(z,y), x) \in \theta$, i.e., $X(z,y) \in \theta_x$. Since the algebra $(Q; \Sigma)$ is idempotent and hyperassociative, the subalgebra $(\theta^*_x; \Sigma)$ will be idempotent and hyperassociative, i.e., the subalgebra $(\theta^*_x; \Sigma)$ is an idempotent, hyperassociative, rectangular subalgebra, and, according to Lemma 9, this algebra is functionally trivial, that is $(\theta^*_x; \Sigma)$ is an idempotent and rectangular semigroup. □

Theorem 2. *Let $(Q; \Sigma)$ is an idempotent and hyperassociative algebra, with the transitive commutativity property. Then it satisfies the following hyperidentity:*

$$Y(X(x,y), X(z,x)) = Y(X(x,z), X(y,x)). \tag{m1}$$

Proof. Consider the relation

$$\theta^* = \{(x,y) \in Q \times Q | X(x, X(y,x)) = x, X(y, X(x,y)) = y, \forall X \in \Sigma\},$$

which, according to Theorem 1, is a congruence. Since $X(x, Y(x, Y(y,x))) = X(Y(x, Y(y,x)), x)$ and $X(x, Y(x, (z,x))) = X(Y(x, Y(z,x)), x)$, the transitivity of commutativity implies

$$X(Y(x, Y(y,x)), Y(x, Y(z,x))) = X(Y(x, Y(z,x)), Y(x, Y(y,x))). \tag{17}$$

Since $(X(x,y), X(x, X(y,x))) \in \theta^*$ and $(X(z,x), X(x, X(z,x))) \in \theta^*$, the fourth condition of the congruence implies

$$(Y(X(x,y), X(z,x)), Y(X(x, X(y,x)), X(x, X(z,x)))) \in \theta^*.$$

In the same way,

$$\begin{cases} (X(x, X(z,x)), X(x,z)) \in \theta^*, \\ (X(x, X(y,x)), X(y,x)) \in \theta^*, \end{cases} \Rightarrow (Y(X(x, X(z,x)), X(x, X(y,x))), Y(X(x,z), X(y,x))) \in \theta^*.$$

According to the transitivity of θ^* and (17), we get

$$(Y(X(x,y), X(z,x)), Y(X(x,z), X(y,x))) \in \theta^*.$$

Thus, since $(Y(X(x,y),X(z,x)),x)\in\theta$ and $(x,Y(X(x,z),X(y,x)))\in\theta$, the transitivity of commutativity implies

$$(Y(X(x,y),X(z,x)),Y(X(x,z),X(y,x)))\in\theta.$$

At last, we get $(Y(X(x,y),X(z,x)),Y(X(x,z),X(y,x)))\in\theta\cap\theta^* \overset{\text{Lemma 5}}{\Rightarrow} Y(X(x,y),X(z,x)) = Y(X(x,z),X(y,x))$. □

Theorem 3. *Let $(Q;\Sigma)$ is an idempotent and hyperassociative algebra with the transitive commutativity property satisfying the following hyperidentity:*

$$Y(X(x,y),Y(z,x)) = Y(X(x,z),Y(y,x)). \tag{m2}$$

Then the relation $\theta=\{(x,y)\in Q\times Q|X(x,y)=X(y,x),\ \forall X\in\Sigma\}$ is a congruence such that the corresponding quotient algebra is a rectangular semigroup, and in each equivalence class any operation of the set Σ is a semilattice operation.

Proof. Let $\theta=\{(x,y)\in Q\times Q|X(x,y)=X(y,x),\ \forall X\in\Sigma\}$. Show, that this relation is a congruence on the algebra $(Q;\Sigma)$, i.e.,

1. $(x,x)\in\theta,\ \forall x\in Q$,
2. $(x,y)\in\theta\Rightarrow(y,x)\in\theta$,
3. $(x,y)\in\theta\&(y,z)\in\theta\Rightarrow(x,z)\in\theta$,
4. $(x,y)\in\theta\&(z,t)\in\theta\Rightarrow(X(x,z),X(y,t))\in\theta,\ \forall X\in\Sigma$.

The definition θ and transitivity of commutativity imply the conditions (1), (2), (3). It remains to check the condition (4).

According to Theorem 2, the algebra $(Q;\Sigma)$ satisfies the hyperidentity $(m1)$. Note, that, according to Lemma 8, the algebra $(Q;\Sigma)$ satisfies the following hyperidentity $X(X(x,y),Y(z,x)) = X(X(x,z),Y(y,x))$, which we will denote by $(m3)$.

We can check the condition (4) using the hyperidentities (as_1), (id), (m1), (m2) and (m3). Therefore θ is a congruence.

Now, show, that the quotient algebra $(Q/\theta;\Sigma/\theta)$ is a rectangular algebra, and in each equivalence class each operation from the set Σ is a semilattice operation. Indeed,

$$X^*(\theta_x,X^*(\theta_y,\theta_x)) = X^*(\theta_x,\theta_{X(y,x)}) = \theta_{X(x,X(y,x))},\quad \forall X^*\in\Sigma/\theta.$$

Show, that $\theta_{X(x,X(y,x))}=\theta_x$. Take $Y\in\Sigma$,

$$Y(X(x,X(y,x)),x) = X(x,Y(X(y,x),x)) = X(x,X(y,Y(x,x)))$$
$$= X(x,X(y,x)) = X(Y(x,x),X(y,x)) = Y(x,X(x,X(y,x)))\Rightarrow\theta_{X(x,X(y,x))}=\theta_x.$$

Therefore $(Q/\theta;\Sigma/\theta)$ is a rectangular algebra. It is easy to check, that the quotient algebra $(Q/\theta;\Sigma/\theta)$ satisfies the hyperidentities (id) and (as_1).

Since $(Q/\theta;\Sigma/\theta)$ is idempotent, hyperassociative and rectangular, then according to Lemma 9, it is an idempotent and rectangular semigroup. Show, that each equivalence class is closed under each operation from the set Σ.

Take $\theta_x\in Q/\theta$, $X\in\Sigma$ and $y,z\in\theta_x$. We have $(y,x)\in\theta$ and $(z,x)\in\theta$, therefore, according to the fourth condition of the congruence, $(X(z,y),X(x,x))\in\theta$, and since $X(x,x)=x$, then $(X(z,y),x)\in\theta$, i.e., $X(z,y)\in\theta_x$.

Now we take $\theta_x\in Q/\theta$ and $y,z\in\theta_x$. We have $y\in\theta_x\&z\in\theta_x\Rightarrow(x,y)\in\theta\&(x,z)\in\theta$, therefore, according to the transitivity of commutativity, $(y,z)\in\theta\Rightarrow X(y,z)=X(z,y),\ \forall X\in\Sigma$, i.e., each equivalence class is closed under each operation from the set Σ. □

Theorem 4. *Let $(Q;\Sigma)$ is an idempotent and hyperassociative algebra. Then, if there exists on $(Q;\Sigma)$ such a congruence, that corresponding quotient algebra is a rectangular semigroup and operations of the set are semilattice operations on each equivalence class, then $(Q;\Sigma)$ satisfies the transitive commutativity property.*

Proof. Let there exist such a congruence θ on the algebra $(Q;\Sigma)$, that the quotient algebra $(Q/\theta;\Sigma/\theta)$ is a rectangular semigroup, and on each equivalence class the operations from the set Σ are semilattice operations, where

$$Q/\theta = \{\theta_x | \theta_x = \{y \in Q | (x,y) \in \theta\}\},$$

$$\Sigma/\theta = \{X^* | X^*(\theta_x, \theta_y) \stackrel{def}{=} \theta_{X(x,y)}\}, \quad X^*(\theta_x, \theta_y) = \theta_x \circ \theta_y, \quad \forall X \in \Sigma.$$

We have

$$\theta_x \circ (\theta_y \circ \theta_x) = \theta_x, \quad X(y,z) = X(z,y), \quad \forall \theta_x \in Q/\theta, \quad \forall y,z \in \theta_x, \quad \forall X \in \Sigma.$$

Let $X(x,y) = X(y,x) \& X(y,z) = X(z,y)$. Prove $X(x,z) = X(z,x)$. Indeed,

$$\theta_x \circ \theta_y = \theta_{X(x,y)} \stackrel{X(x,y)=X(y,x)}{=} \theta_{X(y,x)} = \theta_y \circ \theta_x \Rightarrow \theta_x \circ \theta_y = \theta_y \circ \theta_x \stackrel{\text{Lemma 9}}{\Rightarrow} \theta_x = \theta_y.$$

In the same way, we can get, that $\theta_y = \theta_z$, therefore $\theta_x = \theta_z$. Since each operation from the set Σ on the equivalence classes is a semilattice operation, it is also commutative, so $X(x,z) = X(z,x)$. $\square$

Note, that the results of Theorem 3 and Theorem 4 were proved in the paper [12] for the semigroups case (see also [13, 10]).

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